# The Synthesis of Expanded Electron Beam (Inner Problem) 

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#### Abstract

The method of formation calculation of intense e-beam emitted by flat cathode at $\rho$-regime is considered. The near-cathode are is investigated in antiparaxial approximation. The e-beam behavior in the zone of acceleration is described by the paraxial equations. The obtained solutions are linked at the border of the areas by the axial distributions of the potential, charge density and by the beam envelope. The calculation example is presented demonstrating the methods capabilities and allowing to describe the flow by the simple analytical expressions.


## Introduction

The calculation of electron beam and its focusing system is a really difficult problem. There are a number of problems in this area that can be solved exactly $[1,2]$. That is why the approximating methods of analysis became widely known: paraxial approximation for description of thin beams with low transversal heterogeneities and near surface method of synthesis (also called antiparaxial) for investigation of peculiarities in the near-emitter zone.

The conditions for application of both approximation methods are realized, for example, during the formation and acceleration of e-beam in the gun operating at the regime of current limitation by the electrons spatial charge (so called $\rho$-regime). The e-beam can be described by the near-surface analysis method at the near-cathode area because this zone is characterized by the high charge density and remarkable transversal heterogeneities. The e-beam is well described by the paraxial equations far away from the cathode.

The possibility of such combined description of axially symmetric e-beam generated by the flat cathode at $\rho$-regime is investigated in this paper. The conditions for the solutions linking at the paraxial approximation and near surface analysis areas bounds are determined. The problem with the analytical ebeam description is considered as an example.

## Paraxial Approximation

Paraxial equation for axially symmetric beam is given by $[3,4]$ :

$$
\begin{equation*}
r^{\prime \prime} \varphi+\frac{r^{\prime} \varphi^{\prime}}{2}+\frac{r \varphi^{\prime \prime}}{4}=\frac{I}{4 \pi \varepsilon_{0} r \sqrt{2 \varphi \cdot e / m}}, \tag{1}
\end{equation*}
$$

where the member at the right part of the equation describes the spatial charge influence.

Here $\varphi$ is an electric potential at the beam axis; $r-$ radial coordinate; $e, m$ - electron charge and mass; $I-$ beam current, and dash means the longitudinal coordinate $z$ derivative. Usually the $\varphi(z)$ distribution is known and the dependence $r(z)$ is to be found out given the $r_{0}$ and $r_{0}{ }^{\prime}$ at some $z=z_{0}$. The inverse problem is also possible - to find $\varphi=\varphi(z)$ function providing required $r=r(z)$ dependence. Either $r_{0}$ and $r_{0}{ }^{\prime}$ or $r_{0}$ and $\varphi_{0}$ can be set in this case.

In electron guns the potential $\varphi$ is usually a unique function of $z$. That is why it is possible to look for solution of eq. (1) in form of $r(z)=r[\varphi(z)]=r(\varphi)$. Considering that $r^{\prime}=\varphi^{\prime} d r / d \varphi, r^{\prime \prime}=\varphi^{\prime \prime} d r / d \varphi=\varphi^{\prime 2} d^{2} r / d \varphi^{2}$ and bringing in the symbol:

$$
A=\frac{I}{4 \pi \varepsilon_{0} \sqrt{2 e / m}}
$$

we can write (1) in a form of:

$$
\begin{equation*}
\left(\frac{r}{4}+\varphi \frac{d r}{d \varphi}\right) \varphi^{\prime \prime}+\left(\frac{1}{2} \frac{d r}{d \varphi}+\varphi \frac{d^{2} r}{d \varphi^{2}}\right) \varphi^{\prime 2}-\frac{A}{r \varphi^{1 / 2}}=0 . \tag{2}
\end{equation*}
$$

Let us consider two cases.
Case I. $\varphi d r / d \varphi+r / 4=0\left(\right.$ or $\left.r \sim \varphi^{1 / 4}\right)$
Then the solution of (2) is given by:

$$
\begin{gather*}
\varphi=\left(C_{1} z+C_{2}\right) ;  \tag{3}\\
r=\frac{2}{C_{1}} \sqrt{\frac{A}{3\left(C_{1} z=C_{2}\right)}}, \tag{4}
\end{gather*}
$$

where $C_{1}$ and $C_{2}$ are constants of integration defined by the condition $z=z_{0}$.

The obtained form of the relations excludes their applicability near the cathodes operating at $\rho$-regime. Indeed, given $z=0$ the conditions $\varphi=0$ and $\varphi^{\prime}=0$ are executed at $C_{2}=0$. But at these conditions $r_{0} \rightarrow \infty$ and $r_{0}{ }^{\prime} \rightarrow \infty$ that has no physical meaning. Nevertheless the solution (4), (3) can be realized for inner problem far away from the cathode.

Case II. $\varphi d r / d \varphi+r / 4 \neq 0$
Let us bring in symbols

$$
\begin{equation*}
F(\varphi)=\frac{\frac{1}{2} \frac{d r}{d \varphi}+\varphi \frac{d^{2} r}{d \varphi^{2}}}{\frac{r}{4}+\varphi \frac{d r}{d \varphi}} \tag{5}
\end{equation*}
$$

$$
G(\varphi)=\frac{A}{r \varphi^{1 / 2}\left(\frac{r}{4}+\varphi \frac{d r}{d \varphi}\right)}
$$

and rewrite (2) in the form of

$$
\begin{equation*}
\varphi^{\prime \prime}+F(\varphi)-G(\varphi)=0 . \tag{6}
\end{equation*}
$$

This equation has the following solution:

$$
\begin{equation*}
z+C_{1}=\int \frac{g(\varphi) d \varphi}{\sqrt{C_{2}+2 \int G\left(\varphi^{\prime}\right) g^{2}\left(\varphi^{\prime}\right) d \varphi^{\prime}}}, \tag{7}
\end{equation*}
$$

where $g(\varphi)=\exp \left[\int F\left(\varphi^{\prime}\right) d \varphi^{\prime}\right], C_{1}$ and $C_{2}$ means the same as before.

In spite of complicated form, the formula (7) allows us to find the consistent solutions of eq. (1). They are called "consistent" because of necessity of preliminary assumption of the dependence $r=r(\varphi)$ form. The $F(\varphi)$ and $G(\varphi)$ are determined from (5) in this case. The solution $\varphi(z)$ is found from the integration of (7). And $r=r[\varphi(z)]=r(z)$ dependence is found next.

Let us consider the dependence

$$
\begin{equation*}
r=r_{0}\left(\varphi / \varphi_{0}\right)^{p}, p=\text { const }, \tag{8}
\end{equation*}
$$

where the accomplishment of $r=r_{0}$ and $\varphi=\varphi_{0}$ given $z=z_{0}$ is already considered.

Then

$$
\begin{aligned}
& F=p(p-1 / 2) / \varphi(p+1 / 4) ; \\
& G=\frac{A \varphi_{0}^{2} p}{r_{0}^{2}(p+1 / 4)} \cdot \frac{1}{\varphi^{2 p+1 / 2}} .
\end{aligned}
$$

The expression (7) is integrated "completely", for example by $p=-0,08856$, and the solution is given as:

$$
\begin{align*}
& r=r_{0} \frac{\varphi_{0}^{0,1}}{\left[4,1 \frac{A}{r_{0}^{2}}\left(z-z_{0}\right)^{2}+\varphi_{0}^{3 / 2}\right]^{0,067}} \\
& \varphi=r_{0} \frac{\left[4,1 \frac{A}{r_{0}^{2}}\left(z-z_{0}\right)^{2}+\varphi_{0}^{3 / 2}\right]^{25 / 33}}{\varphi_{0}^{0,134}} \tag{9}
\end{align*}
$$

Apparently this solution can be applied only at sufficient remoteness from the cathode.

## Near Surface Analyses

Hydrodynamic model of intense electron flow in the absence of magnetic field and under assumption of its regularity is described by the system [1]:
$\operatorname{rot}(v)=0, d v / d t=\operatorname{grad}(\varphi), \operatorname{div}(\rho v)=0, \rho=\Delta \varphi$,
where a number of constants are omitted for the simplification (particle charge and mass ( $e, m$ ), dielectric constant $\varepsilon_{0}$ ).

Let us bring in the movement potential $S$, defined as:

$$
\begin{equation*}
V=\operatorname{grad}(S) \tag{11}
\end{equation*}
$$

The system (10) allows looking for solution in the neighborhoods of flat equipotent cathode operating at $\rho$-regime in form of following rows:

$$
\left\{\begin{array}{l}
S=f_{1} z^{5 / 3}+f_{2} z^{11 / 3}+f_{3} z^{17 / 3}  \tag{12}\\
v_{r}=\partial S / \partial r, v_{r}=\partial S / \partial z \\
\varphi=2 v^{2}, v^{2}=v_{r}^{2}+v_{z}^{2} \\
\rho=\frac{\partial^{2} \varphi}{\partial z^{2}}+\frac{1}{r} \frac{\partial \varphi}{\partial r}+\frac{\partial^{2} \varphi}{\partial r^{2}} \\
\frac{\partial\left(\rho v_{z}\right)}{\partial z}+\frac{\rho v_{r}}{r}+\frac{\partial\left(\rho v_{r}\right)}{\partial r}=0
\end{array}\right.
$$

Here $f_{i}=f_{i}(r), i=1,2,3$ are functions of radial coordinates. In fact, the form of dependence $f_{1}(r)$ determines the beam parameters, because it uniquely defines the form of functions $f_{2}(r), f_{3}(r)$. The substitution of $f_{1}(r)$ into (12) gives:

$$
\begin{align*}
& f_{2}=-\frac{1}{132}\left(19 f_{1}^{\prime 2} / f_{1}+8 f_{1}^{\prime \prime}+8 f_{1}^{\prime} / r\right) ;  \tag{13}\\
& f_{3}=-\frac{28072}{23800} \frac{f_{2}^{2}}{f_{1}}\left(1+0,083856 \frac{f_{1}^{\prime 2}}{f_{1} f_{2}}+\right. \\
& +0,164078 \frac{f_{1}^{\prime} f_{2}^{\prime}}{f_{2}^{2}}+0,074451 \frac{f_{1}^{\prime \prime}}{f_{2}}+ \\
& +0,074451 \frac{f_{1}^{\prime}}{r f_{2}}+0,023867 \frac{f_{1} f_{2}^{\prime}}{r f_{2}^{2}}+ \\
& +0,023867 \frac{f_{1} f_{2}^{\prime \prime}}{f_{2}^{2}}+0,013145 \frac{f_{1}^{\prime 2} f_{2}^{\prime \prime}}{f_{1} f_{2}^{2}}+  \tag{14}\\
& +0,0086856 \frac{f_{1}^{\prime} f_{1}^{\prime \prime}}{r f_{2}^{2}}+0,00545 \frac{f_{1}^{\prime} f_{1}^{\prime \prime \prime}}{f_{2}^{2}}+ \\
& \left.+0,00545 \frac{f_{1}^{\prime 3}}{r f_{1} f_{2}^{2}}\right) .
\end{align*}
$$

Every following $f_{i}(r)$ is evaluated by $f_{i-1}(r), f_{i-2}(r) \ldots$ They forms become more bulky with $i$ growth. Knowledge of $f_{1}(r)$ and $f_{2}(r)$ is enough for the description of the near surface area. Knowledge of $f_{3}(r)$ can be used for the determination of area extension where the neglect of other expansion terms is true.

The following e-beam parameters can be determined with the help of $f_{i}(r)$ :

- potential distribution:

$$
\begin{align*}
& \varphi=\frac{25}{18} f_{1}^{2} z^{4 / 3}+\left(\frac{1}{2} f_{1}^{\prime 2}+\frac{50}{9} f_{1} f_{2}\right) z^{10 / 3}+ \\
& +\left(f_{1}^{\prime} f_{2}^{\prime}+\frac{121}{18} f_{2}^{2}+\frac{85}{9} f_{1} f_{3}\right) z^{16 / 3} \tag{15}
\end{align*}
$$

- density:

$$
\begin{align*}
& \rho=\frac{50}{81} \frac{f_{1}^{2}}{z^{2 / 3}}+\left(\frac{20}{3} f_{1}^{2}+\frac{3850}{81} f_{1} f_{2}+\frac{25}{9} f_{1} f_{1}^{\prime \prime}+\right. \\
& \left.\frac{25}{9} \frac{f_{1} f_{1}^{\prime}}{r}\right) z^{4 / 3}+\left(\frac{106}{3} f_{1}^{\prime} f_{2}^{\prime}+\frac{12584}{81} f_{2}^{2}+\right. \\
& +\frac{17680}{81} f_{1} f_{3}+f_{1}^{\prime \prime 2}+f_{1}^{\prime} f_{1}^{\prime \prime \prime}+\frac{55}{9} f_{1}^{\prime \prime} f_{2}+ \\
& \left.+\frac{55}{9} f_{1} f_{2}^{\prime \prime}+\frac{f_{1}^{\prime} f_{1}^{\prime \prime}}{r}+\frac{55}{9} \frac{f_{1}^{\prime} f_{2}}{r}+\frac{55}{9} \frac{f_{1} f_{2}}{r}\right) z^{10 / 3}, \tag{16}
\end{align*}
$$

- and envelope:

$$
\begin{equation*}
\frac{d r}{d z}=\frac{v_{r}}{v_{z}}=\frac{\partial S / \partial r}{\partial S / \partial z}=z \frac{f_{1}^{\prime}+f_{2}^{\prime} z^{2}+f_{3}^{\prime} z^{4}}{\frac{5}{3} f_{1}+\frac{11}{3} f_{2} z^{2}+\frac{17}{3} f_{3} z^{4}} \tag{17}
\end{equation*}
$$

with initial condition that $r=r_{0}$ given $z=0$.
The linking of solutions obtained in paraxial and antiparaxial approximations consists in e-beam parameters aligning in some point with coordinate $z=z_{0}$. This point position (which is by the way calculates itself) should satisfy following conditions: 1) the condition of near surface analysis application (i.e. the neglect of expansion terms born by $f_{3}(r)$ and proportional to $z^{4}$ ) is true; 2) the e-beam potential, density, envelope and its tangent to the axis $r_{0}{ }^{\prime}$ should coincide for both approximations.

Let us explain written before on example of converging e-beam formation emitted by the flat cathode.

## Electron Beam Calculation

Let us consider the problem allowing the obtaining of final solution in form of relatively simple analytical expressions. Let us choose the solution for the paraxial equation in form of:

$$
\begin{gather*}
\varphi=b(z+D)^{2}  \tag{18}\\
\rho=\beta(z+D)^{-0,5} \tag{19}
\end{gather*}
$$

where $\beta, b$ and $D$ are some constants.
The charge density in such e-beam is constant and equal to

$$
\begin{equation*}
\rho=2 b \tag{20}
\end{equation*}
$$

and the envelope tangent changes as

$$
\begin{equation*}
r^{\prime}=-\beta(\mathrm{z}+D)^{-3 / 2} / 2 . \tag{21}
\end{equation*}
$$

For the description of the near-surface area let us choose as $f_{1}$ the expression corresponding to the ebeam emitted by the flat cathode perpendicular to the surface and experiencing focusing by the static electric field (the magnetic field is absent):

$$
\begin{equation*}
f_{1}(r)=\left(1-r^{2} / a^{2}\right)^{2} \tag{22}
\end{equation*}
$$

here $a$ is the e-beam transversal heterogeneity parameter. From (13) and (14) we find:

$$
\begin{gather*}
f_{2}=\frac{8}{33 a^{2}}\left(1-\frac{27}{8} \frac{r^{2}}{a^{2}}\right)  \tag{23}\\
f_{3}=\frac{0,04141}{a^{4}}\left(1-1,9 \frac{r^{2}}{a^{2}}\right) . \tag{24}
\end{gather*}
$$

Substituting $f_{1}, f_{2}, f_{3}$ into (15), (16), (17) we find:

$$
\begin{aligned}
& \varphi=\frac{25}{18} z^{4 / 3}\left[1-2 \frac{r^{2}}{a^{2}}+\frac{32}{33} \frac{z^{2}}{a^{2}}\left(1-2,89 \frac{r^{2}}{a^{2}}\right)+\right. \\
& \left.+0,567 \frac{z^{4}}{a^{4}}\left(1+1,463 \frac{r^{2}}{a^{2}}\right)\right] ; \\
& \rho=\frac{50}{81 z^{2 / 3}}\left[1-2 \frac{r^{2}}{a^{2}}+\frac{2}{3} \frac{z^{2}}{a^{2}}\left(1+3,7 \frac{r^{2}}{a^{2}}+\right.\right. \\
& \left.\left.+\frac{189}{20} \frac{r^{4}}{a^{4}}\right)+0,485 \frac{z^{4}}{a^{4}}\left(1-65 \frac{r^{2}}{a^{2}}\right)\right] ; \\
& \frac{d r}{d z}=-\frac{6}{5} \frac{r z}{a^{2}} \frac{1+\frac{9}{11}}{1-\frac{r^{2}}{a^{2}}+\frac{8}{15} \frac{z^{2}}{a^{2}}-\frac{r^{2}}{2 a^{2}}-\frac{9}{5} \frac{r^{2}}{a^{2}} \frac{z^{2}}{a^{2}}}
\end{aligned}
$$

Following calculations are carried out assuming that transversal heterogeneity is small enough, i.e. that difference between the current density at the cathode edge and center is less then $30 \%$. In other words, following conditions are have to be fulfilled given that $z=0$ :

$$
\left.\rho v_{z}\right|_{z=0} ^{r=v_{0}}=0,\left.7 \rho v_{z}\right|_{\substack{r=0 \\ z=0}} ^{\substack{0}}
$$

where

$$
\left.\rho v_{z}\right|_{z=z_{0}}=\frac{250}{243} f_{1}^{3} .
$$

Then

$$
\begin{equation*}
\frac{r_{0}^{2}}{a^{2}}=\frac{1}{20} . \tag{25}
\end{equation*}
$$

Under made assumptions from (15)-(17) we can find main distributions:

$$
\begin{gather*}
\varphi=\frac{25}{18} z^{4 / 3}\left(1+\frac{32}{33} \frac{z^{2}}{a^{2}}+0,567 \frac{z^{4}}{a^{4}}\right) ;  \tag{26}\\
\rho=\frac{50}{81 z^{2 / 3}}\left(1+\frac{2}{3} \frac{z^{2}}{a^{2}}+0,485 \frac{z^{4}}{a^{4}}\right) ;  \tag{27}\\
r=r_{0}\left(1-\frac{3}{5} \frac{z^{2}}{a^{2}}\right) . \tag{29}
\end{gather*}
$$

Comparing (26)-(28) with (18)-(21) we can find:

$$
\left\{\begin{array}{l}
z_{0}=0,415 a ;  \tag{29}\\
D=0,4852 a \\
b=0.626 / a^{2 / 3} \\
\beta=0,85 r_{0} \sqrt{a}
\end{array}\right.
$$

So in near-cathode area $\left(z<z_{0}\right)$ the e-beam parameters are described by the expressions (26)-(28) with constants (29), and with $z>z_{0}$

$$
\begin{gathered}
\varphi=\frac{0,625}{a^{2 / 3}}(z+0,485 a)^{2} ; \\
\rho=1,25 / a^{2 / 3} ; \\
r=\frac{0,85 r_{0} \sqrt{a}}{\sqrt{z+0,485 a}} .
\end{gathered}
$$

The e-beam envelope main axial distributions of potential and density are represented in Figs. 1-3.


Fig. 1. E-beam envelope


Fig. 2. E-beam potential distribution


Fig. 3. E-beam spatial charge density distribution

## Conclusion

The method of formation calculation of intense ebeam emitted by flat cathode at $\rho$-regime is considered. The near-cathode is investigated in antiparaxial approximation. The e-beam behavior in the zone of acceleration is described by the paraxial equations. The obtained solutions are linked at the border of the areas by the axial distributions of the potential, charge density and by the beam envelope. The calculation example is presented demonstrating the methods capabilities and allowing to describe the flow by the simple analytical expressions.

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