# Equilibrium Configurations of Uncharged Conducting Jets in a Transverse Electric Field ${ }^{1}$ 

O.V. Zubareva, N.M. Zubarev<br>Institute of Electrophysics, UB, RAS,Amundsen st., 106, Ekaterinburg,620016, Russia<br>Phone: (343)2678776, Fax: (343)2678794,E-mail: olga@ami.uran.ru


#### Abstract

Solutions of the problem concerning equilibrium configurations of an uncharged conducting liquid jet in a transverse electric field have been obtained. These solutions correspond to cylindrical jets with the cross section strongly deformed in the direction of field. The range of the electric field strength values is determined, where the stable configurations exist. The problem does not admit solutions for the field strengths above the threshold that corresponds to the jet splitting.


## 1. Introduction

In the absence of an external electric field, the only stable equilibrium jet configuration is a round cylinder. Let us consider a jet of conducting fluid in an electric field, which is oriented perpendicularly to the jet axis. Electrostatic force leads to the azimuthal deformation of the jet surface: its cross section will be stretched along the field. Compensation of the electrostatic forces by the surface tension will lead to a new equilibrium configuration of the jet surface.

As is known, capillary forces (in the absence of electric fields) lead to the development of the socalled Rayleigh instability of a round cylindrical jet, which is manifested by the growth of longitudinal perturbations with a characteristic wavelength exceeding the circumference length [1]. Similar analysis of the stability of a jet placed into an electric field is hindered by the fact that the unperturbed solution for the jet shape is unknown (it differs from the round cylinder). Therefore, the necessity arises to investigate possible equilibrium configurations of jets in a transverse electric field. Our aim is to find the equilibrium configurations of a jet strongly deformed and to determine a range of electric field strength, where the solutions exist.

Recently, an exact particular solution was found [2] for the special case where the difference of pressures inside and outside the jet is zero. This solution corresponds to a jet strongly deformed, whereby the aspect ratio of the jet cross section $(A)$ amount to 23/4. It should be noted that the similar solution was ob-
tained by McLeod [3] for a mathematically analogous problem of finding the shape of a two-dimensional gas bubble moving in the ideal liquid and then was studied in detail [4]. As was demonstrated in [2], this solution is unstable with respect to small azimuthal deformations for the jet shape and is of no physical interest (stable configurations correspond to deformations leading to the aspect ratios significantly below 23/4). Nevertheless, this solution can be useful in searching approximate solutions of the problem. It is related with our demands to the approximations of the surface shape. The solutions must be exact in two cases: the case, when an external field is absent (round jet, $A=1$ ) and the case, when a difference of the pressure at the jet boundary is zero ( $A=23 / 4$ ). This condition allows us to construct "almost" exact interpolation solution for jet configurations in the range of $1<A<23 / 4$.

## 2. Initial equations

Let us assume that the jet cross section remains constant and the fluid is in rest in the frame of reference moving with the jet (plane symmetry of the problem). Let us write the equations of electrostatics that describe a stationary profile of the uncharged surface of the conducting liquid jet in the transverse electric field of the strength $E$. The electric field potential $\varphi$ in the jet cross section plane $\{x, y\}$ is described by the twodimensional Laplace equation:

$$
\varphi_{x x}+\varphi_{y y}=0 .
$$

This equation has to be solved together with the equipotentiality condition $\varphi=0$ for the conductor surface.
Another boundary condition corresponds to homogeneity of the electric field at infinity.

$$
\begin{equation*}
\varphi \rightarrow-E y, \quad x^{2}+y^{2} \rightarrow \infty . \tag{1}
\end{equation*}
$$

The equilibrium relief of the surface of a conducting fluid is determined by the condition of balance for the forces acting on this charged surface:

$$
\begin{equation*}
(8 \pi)^{-1}(\nabla \varphi)_{\varphi=0}^{2}+T K+P=0, \tag{2}
\end{equation*}
$$

where the first term describes the electrostatic pressure at the liquid boundary and the second term describes

[^0]the surface pressure ( $T$ is the surface tension coefficient and $K$ is the local curvature of the surface). For the surface given by the parametric expression
$$
y=Y(\tau), \quad x=X(\tau),
$$
where $\tau$ is a parameter (we will define it below), the curvature is determined by the expression
$$
K=\frac{X_{\tau} Y_{\tau \tau}-Y_{\tau} X_{\tau \tau}}{\left(X_{\tau}^{2}+Y_{\tau}^{2}\right)^{3 / 2}} .
$$

The constant $P$ gives a difference of pressures inside and outside the jet. In the absence of electric field, a jet has a round cross section and this parameter is given by the expression $P=P_{0} \equiv T \sqrt{\pi / S}$, where $S$ is a cross section area of the jet. A comparison of Eq.(1) with equations describing the shape of a gas bubble moving in the ideal liquid [3] shows that they coincide after the following substitutions:

$$
E \rightarrow-\sqrt{4 \pi \rho} V, \quad \varphi \rightarrow \sqrt{4 \pi \rho} \Psi
$$

where $V$ is the velocity of liquid streamlining the bubble, $\rho$ is the density of this liquid, and $\Psi$ is the current function. Rewritten in terms of these variables, the balance condition for the forces acting on the surface (2) becomes the stationary Bernoulli equation for the bubble surface, in which constant $P$ describes a difference between the pressure inside the bubble and the energy density in the liquid at infinity. The particular solution obtained by McLeod [3] corresponds to the case of $P=0$. As applied to the problem of the jet geometry in electrical field, this solution is discussed in our paper [2].

For convenience, we convert to the dimensionless variables

$$
\begin{array}{ll}
x \rightarrow\left(4 \pi \lambda T / E^{2}\right) x, & y \rightarrow\left(4 \pi \lambda T / E^{2}\right) y, \\
\varphi \rightarrow(4 \pi \lambda T / E) \varphi, & P \rightarrow\left(E^{2} /(4 \pi \lambda)\right) p,
\end{array}
$$

where $\lambda$ is a constant, the value of which we choose solving the problem.

Let us introduce a so-called complex potential $w=\psi-i \varphi$, which is an analytic function of the complex variable $z=x+i y$. The function $\psi$ is conjugate harmonic function with the electric field potential $\varphi$; the condition $\psi=$ const defines the electric field lines. The potential $w$, as follows from (1), satisfies the condition at infinity

$$
w \rightarrow z, \quad|z| \rightarrow \infty,
$$

and the balance condition for the forces acting on the jet surface (2) can be written as

$$
\frac{\lambda}{2}\left|\frac{d w}{d z}\right|^{2}+K+p=0
$$

As the next step solving of our problem we deduce the conformal mapping the exterior of the jet to the exterior of the unit circle $(z \rightarrow \xi)$. The problem of finding the complex potential under the condition $\operatorname{Im} w=0$ on an unknown surface $z=Z(\tau)$, where
$Z=X+i Y$, reduces to the problem with analogous condition on the unit circle $|\xi|=1$. It is identical to the known problem of plane potential flow past a round cylinder in the incompressible fluid. The solution of the problem is given as

$$
\begin{equation*}
w=\xi+\frac{1}{\xi} \tag{3}
\end{equation*}
$$

Thus our problem reduces to finding an unknown analytic function $z=z(\xi)$, which satisfies two conditions:

$$
\begin{gather*}
\frac{\lambda}{2} \frac{\left|1-\xi^{-2}\right|^{2}}{\left|z_{\xi}\right|^{2}}-\frac{\operatorname{Re}\left(\xi z_{\xi \xi} z_{\xi}\right)+\left|z_{\xi}\right|^{2}}{\left|z_{\xi}\right|^{3}}+p=0, \quad|\xi|=1,  \tag{4}\\
z \rightarrow \xi, \quad|\xi| \rightarrow \infty . \tag{5}
\end{gather*}
$$

Taking into consideration (5), $z$ should be expanded in the form

$$
z(\xi)=\xi+\frac{a_{1}}{\xi}+\frac{a_{2}}{\xi^{3}}+\ldots
$$

The absence of even powers is connected with the problem symmetry.

As is evident from [4], the substitution of such series to (4) leads to rather complicated relations, which are inconvenient to use. First of all this is related with the availability of fractional degrees $\left|z_{\xi}\right|^{3}=\left(z_{\xi} z_{\xi}\right)^{3 / 2}$ in denominator of second term of equation (4). One way to get around this problem is to introduce a new function

$$
\begin{align*}
& g(\xi)=\sqrt{i \xi z_{\xi}}=\sqrt{i \xi}\left(1+\frac{\alpha_{1}}{\xi^{2}}+\frac{\alpha_{2}}{\xi^{4}}+\ldots\right),  \tag{6}\\
& \alpha_{1}=-a_{1} / 2, \quad \alpha_{2}=-3 a_{2} / 2-a_{1}^{2} / 8, \quad \ldots
\end{align*}
$$

Using $g(\xi)$ we rewrite the boundary condition (4) in the following form:

$$
\begin{equation*}
\frac{\lambda}{2}\left|1-\xi^{-2}\right|^{2}-2 \operatorname{Re}\left(\bar{g} g_{\xi} \xi\right)+p(g \bar{g})^{2}=0 . \tag{7}
\end{equation*}
$$

It should be noted that a similar equation for a bubble in two-dimensional flow was obtained by Tanveer [5]).

The circle $|\xi|=1$, which corresponds to jet surface in the term of initial variables, can be parameterized as $\xi=e^{i \tau}$, where $\tau$ is a real parameter. Then the jet surface is given by the expression:

$$
z=z\left(e^{i \tau}\right) \equiv Z(\tau),
$$

where $0 \leq \tau<2 \pi$. It is convenient also to introduce a complex function $G(\tau) \equiv g\left(e^{i \tau}\right)$, which determines a value of the analytic function $g$ at the boundary $|\xi|=1$. It is connected with the function $Z$ by simple relation: $G(\tau)=\sqrt{d Z / d \tau}$. Using this function we rewrite Eq.(11) in the following form:

$$
\begin{equation*}
\frac{\lambda}{2}\left(2-e^{-2 i \tau}-e^{2 i \tau}\right)+i\left(\bar{G} G_{\tau}-G \bar{G}_{\tau}\right)+p(G \bar{G})^{2}=0 \tag{8}
\end{equation*}
$$

This equation with the fourth-order nonlinearity will be an object of our following consideration.

## 3. The problem solutions

In terms of the definition (6), the solution of equation (8) will have infinite series representation

$$
\begin{equation*}
G(\tau)=e^{i \pi / 4+i \tau / 2}\left(1+\alpha_{1} e^{-2 i \tau}+\alpha_{2} e^{-4 i \tau}+\ldots\right) \tag{9}
\end{equation*}
$$

As noted above, two exact solutions are known for the discussed problem. The series is finite for the both solutions. Thus, for the jet of round cross section in the absent of electrical fields ( $p=1$ and $\lambda=0$ ) we have

$$
\begin{equation*}
G(\tau)=e^{i \pi / 4+i \tau / 2} \tag{10}
\end{equation*}
$$

The other case exactly solved corresponds to $p=0$ and $\lambda=2 / 3$. For these parameter values the equation (8) is essentially simplified. It has become squarenonlinear. As can be easily verified, the next expression will be its solution

$$
\begin{equation*}
G(\tau)=e^{i \pi / 4+i \tau / 2}\left(1+\frac{1}{3} e^{-2 i \tau}\right) \tag{11}
\end{equation*}
$$

This particular solution, which corresponds to the strongly deformed jet, is unstable with respect to small azimuth perturbations [2].

Possible stationary configurations of bubbles have been investigated in a range of values of the parameters $p$ and $\lambda$ which are close to $p=1$ and $\lambda=0$, and also to $p=0$ and $\lambda=2 / 3$ [4]. Asymptotic expansions in the form of (9) in the neighborhood of the appropriate exact solutions have been constructed. Their analysis showed that the use of a sufficiently large number of terms in the expansion (9) permits to describe the surface configurations of the bubbles over a wide range of parameter values of the problem with a high accuracy. From our point of view, the main disadvantage of such expansions is that they cannot in principle include both known exact solutions (10) and (11). Moreover, the use of the complicated expressions for jet configurations makes the analytic investigation, in particular the investigation of their stability, much more difficult.

Confining the number of terms of the series (9), and using no restrictions for values of the parameters $p$ and $\lambda$, we seek an approximate solution of the equation (8) in the following form:

$$
\begin{equation*}
G(\tau)=e^{i \pi / 4+i / 2}\left(1+\alpha e^{-2 i \tau}\right) \tag{12}
\end{equation*}
$$

In spite of its simple form, this representation allows to find an "almost" exact solution of the problem. The solution becomes exact at $\alpha=0$ and at $\alpha=1 / 3$, where it coincides with (10) or (11). Of cause, one could seek the problem solution taking into account higher harmonics of the expansion (9). But all physical interesting solutions of the problem are found to lie in the interval $0<\alpha<1 / 3$ between two exact solutions. This means that we construct the interpolated
solution. As an error of interpolation is always less than an error of extrapolation, the expression (12) yields the sufficiently accurate approximation of the solution of Eq.(8).


Fig. 1. The relative approximation error $f$ as a function of the parameter $\alpha$

Substituting the expression (12) into (8), we obtain:

$$
\begin{align*}
& \lambda(1-\cos (2 \tau))-\left(1-3 \alpha^{2}-2 \alpha \cos (2 \tau)\right)+2 p \alpha^{2} \cos (4 \tau) \\
& \quad+p\left(\left(1+\alpha^{2}\right)^{2}+2 \alpha^{2}+4 \alpha\left(1+\alpha^{2}\right) \cos (2 \tau)\right)=0 \tag{13}
\end{align*}
$$

We will neglect the third term (that is, the fourth harmonic) in this expression. Equating the coefficients for the identical harmonics we obtain the equations connecting parameters $\lambda, p$, and $\alpha$ :

$$
\begin{gathered}
\lambda-1+3 \alpha^{2}+p+4 p \alpha^{2}+p \alpha^{4}=0 \\
-\lambda / 2+\alpha+2 p \alpha\left(1+\alpha^{2}\right)=0
\end{gathered}
$$

Then the parameters $\lambda$ and $p$ can be expressed in terms of $\alpha$ :

$$
\begin{align*}
& p(\alpha)=\frac{1-2 \alpha-3 \alpha^{2}}{1+4 \alpha+4 \alpha^{2}+4 \alpha^{3}+\alpha^{4}} \\
& \lambda(\alpha)=\frac{2 \alpha\left(3-5 \alpha^{4}\right)}{1+4 \alpha+4 \alpha^{2}+4 \alpha^{3}+\alpha^{4}} \tag{14}
\end{align*}
$$

These relations, together with (12) give the desired approximate solution of the problem.

Let us estimate the term, which was omitted. Its maximal value is seen to be equal to $2 p \alpha^{2}$. We can compare this term, for example, with the amplitude of the second harmonic, which is equal to $\lambda$ in the expression (13). Their quotient gives a relative approximation error $f$ :

$$
f(\alpha)=\frac{2 p(\alpha) \alpha^{2}}{\lambda(\alpha)}=\frac{\alpha\left(1-2 \alpha-3 \alpha^{2}\right)}{3-5 \alpha^{4}}
$$

As is obvious from the plot (Fig. 1), in the interval $0<\alpha<1 / 3$, it is $\simeq 0.03$ in maximum. We believe that this accuracy is sufficient to restrict ourselves by the representation (12) for the function $G$ for constructing the equilibrium solutions for the jet configurations.


Fig. 2. The cross sections of the jet for (a) $\alpha=0$, (b)

$$
\alpha=0.234, \text { (c) } \alpha=1 / 3, \text { (d) } \alpha=-3+2 \sqrt{3}
$$

Let us find the surface configurations corresponding to the expression (12). Since $d Z / d \tau=G^{2}$, we obtain by integration $\tau$ :

$$
\begin{equation*}
Z(\tau)=e^{i \tau}-2 \alpha e^{-i \tau}-\frac{\alpha^{2}}{3} e^{-3 i \tau} \tag{15}
\end{equation*}
$$

Separating real part from the imaginary one in equation $z=Z(\tau)$, we get the desired parametric expressions for equilibrium surface:

$$
\begin{aligned}
& x=X(\tau)=(1-2 \alpha) \cos (\tau)-\frac{\alpha^{2}}{3} \cos (3 \tau) \\
& y=Y(\tau)=(1+2 \alpha) \sin (\tau)+\frac{\alpha^{2}}{3} \sin (3 \tau)
\end{aligned}
$$

where $\tau$ changes in the range $0 \leq \tau \leq 2 \pi$. The parameter $\alpha$ satisfies the condition $0 \leq \alpha \leq-3+2 \sqrt{3}$. Its maximal value corresponds to surface with selfintersection (see Fig. 2); it can be found from the condition $Z=0$. The solutions have no physical meaning for the greater values of $\alpha$.

## 4. Conclusion

In the present work, we have obtained the oneparameter family of the approximate solutions of the classical problem in electrostatics, namely the problem of finding the equilibrium configuration of an uncharged jet of a conducting liquid in the transversal
electric field. The approach applied is based on the conformal mapping of the region outside of jet to the region outside of unit circle, which restricts the configurations considered by us to the case of the plane symmetry of the problem. The solutions found correspond to the azimuthal deformations of the jet surface under the action of electrostatic forces whereas longitudinal deformations have not been considered. Our analysis allows us to define a threshold value of external field,

$$
E_{c}^{2} \approx 7.02 P_{0} \approx 12.4 T S^{-1 / 2}
$$

depending on the surface tension coefficient $T$ and the jet cross-section aria $S$ (this value corresponds to $\alpha \approx 0.234$ ). If the electric field strength exceeds its critical value, the equilibrium configurations of the jet surface do not exist and, consequently, the jet splits.

It can be noted that the investigation of possible instabilities of the jet in the external electric field requires knowledge of its unperturbed configuration. In the case of a relatively small field, $E \ll E_{c}$, this configuration is close to the round cylinder, and the analysis can be performed in the framework of the theory of perturbation with respect to small deformations of the round jet surface. For a sufficiently strong field equilibrium jet configuration differs from the cylinder with round cross-section and, for the stability analysis, it is necessary to use the solution that was obtained by nonperturbative methods. We have found the corresponding solutions in the present work.

## References

[1] Lord Rayleigh, Proc. London Math. Soc. 10, 4 (1878).
[2] N.M. Zubarev, O.V. Zubareva, Tech. Phys. Lett. 31, 862 (2005).
[3] E.B. McLeod, J. Rat. Mech. Anal. 4, 557 (1955)
[4] P.N. Shankar, J. Fluid Mech. 244, 187 (1992).
[5] S. Tanveer, Proc. R. Soc. Lond. A. 452, 1397 (1996).


[^0]:    ${ }^{1}$ The work was performed within the framework of the program "Mathematical Method in Nonlinear dynamics" of the Presidium of the Russian Academy of Sciences and the Interdisciplinary program for support of the projects from UB RAS and SB RAS. It was financially supported by the Foundation for Support of Russian Science and by RosNauka (contract no. 02.442.11.7242).

